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GLOBAL ASPECTS OF CURRENT ALGEBRA

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A new mathematical framework for the Wess-Zumino chiral effective action is described. It is shown that this action obeys an a priori quantization law, analogous to Dirac's quantization of magnetic change. It incorporates in current algebra both perturbative and non-perturbative anomalies.

The purpose of this paper is to clarify an old but relatively obscure aspect of current algebra: the Wess-Zumino effective lagrangian [1] which summarizes the effects of anomalies in current algebra. As we will see, this effective lagrangian has unexpected analogies to some 2 + 1 dimensional models discussed recently by Deser et al. [2] and to a recently noted SU(2) anomaly [3]. There also are connections with work of Balachandran et al. [4].

For definiteness we will consider a theory with $SU(3)_L \times SU(3)_R$ symmetry spontaneously broken down to the diagonal SU(3). We will ignore explicit symmetry-breaking perturbations, such as quark bare masses. With $SU(3)_L \times SU(3)_R$ broken to diagonal SU(3), the vacuum states of the theory are in one to one correspondence with points in the SU(3) manifold. Correspondingly, the low-energy dynamics can be conveniently described by introducing a field $U(x^\alpha)$ that transforms in a so-called non-linear realization of $SU(3)_L \times SU(3)_R$. For each space-time point x^α, $U(x^\alpha)$ is an element of SU(3): a 3×3 unitary matrix of determinant one. Under an $SU(3)_L \times SU(3)_R$ transformation by unitary matrices (A, B), U transforms as $U \to AUB^{-1}$.

The effective lagrangian for U must have $SU(3)_L \times SU(3)_R$ symmetry, and, to describe correctly the low-energy limit, it must have the smallest possible number of derivatives. The unique choice with only two derivatives is

$$\mathcal{L} = \tfrac{1}{16} F_\pi^2 \int \mathrm{d}^4x \, \mathrm{Tr}\, \partial_\mu U \, \partial_\mu U^{-1} , \tag{1}$$

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where experiment indicates $F_\pi \simeq 190$ MeV. The perturbative expansion of U is

$$U = 1 + \frac{2i}{F_\pi} \sum_{a=1}^{8} \lambda^a \pi^a + \cdots , \tag{2}$$

where λ^a (normalized so $\mathrm{Tr}\,\lambda^a\lambda^b = 2\delta^{ab}$) are the SU(3) generators and π^a are the Goldstone boson fields.

This effective lagrangian is known to incorporate all relevant symmetries of QCD. All current algebra theorems governing the extreme low-energy limit of Goldstone boson S-matrix elements can be recovered from the tree approximation to it. What is less well known, perhaps, is that (1) possesses an extra discrete symmetry that is *not* a symmetry of QCD.

The lagrangian (1) is invariant under $U \leftrightarrow U^{\mathrm{T}}$. In terms of pions this is $\pi^0 \leftrightarrow \pi^0$, $\pi^+ \leftrightarrow \pi^-$; it is ordinary charge conjugation. (1) is also invariant under the naive parity operation $\boldsymbol{x} \leftrightarrow -\boldsymbol{x}$, $t \leftrightarrow t$, $U \leftrightarrow U$. We will call this P_0. And finally, (1) is invariant under $U \leftrightarrow U^{-1}$. Comparing with eq. (2), we see that this latter operation is equivalent to $\pi^a \leftrightarrow -\pi^a$, $a = 1, \ldots, 8$. This is the operation that counts modulo two the number of bosons, N_{B}, so we will call it $(-1)^{N_{\mathrm{B}}}$.

Certainly, $(-1)^{N_{\mathrm{B}}}$ is not a symmetry of QCD. The problem is the following. QCD is parity invariant only if the Goldstone bosons are treated as pseudoscalars. The parity operation in QCD corresponds to $\boldsymbol{x} \leftrightarrow -\boldsymbol{x}$, $t \leftrightarrow t$, $U \leftrightarrow U^{-1}$. This is $P = P_0(-1)^{N_{\mathrm{B}}}$. QCD is invariant under P but not under P_0 or $(-1)^{N_{\mathrm{B}}}$ separately. The simplest process that respects all bona fide symmetries of QCD but violates P_0 and $(-1)^{N_{\mathrm{B}}}$ is $\mathrm{K}^+\mathrm{K}^- \to \pi^+\pi^0\pi^-$ (note that the ϕ meson decays to both $\mathrm{K}^+\mathrm{K}^-$ and $\pi^+\pi^0\pi^-$). It is natural to ask whether there is a simple way to add a higher-order term to (1) to obtain a lagrangian that obeys *only* the appropriate symmetries.

The Euler-Lagrangian equation derived from (1) can be written

$$\partial_\mu\left(\tfrac{1}{8}F_\pi^2 U^{-1}\partial_\mu U\right) = 0 . \tag{3}$$

Let us try to add a suitable extra term to this equation. A Lorentz-invariant term that violates P_0 must contain the Levi-Civita symbol $\varepsilon_{\mu\nu\alpha\beta}$. In the spirit of current algebra, we wish a term with the smallest possible number of derivatives, since, in the low-energy limit, the derivatives of U are small. There is a unique P_0-violating term with only four derivatives. We can generalize (3) to

$$\partial_\mu\left(\tfrac{1}{8}F_\pi^2 U^{-1}\partial_\mu U\right) + \lambda\varepsilon^{\mu\nu\alpha\beta}U^{-1}(\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}(\partial_\alpha U)U^{-1}(\partial_\beta U) = 0 , \tag{4}$$

λ being a constant. Although it violates P_0, (4) can be seen to respect $P = P_0(-1)^{N_{\mathrm{B}}}$.

Can eq. (4) be derived from a lagrangian? Here we find trouble. The only pseudoscalar of dimension four would seem to be $\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,U^{-1}(\partial_\mu U)\cdot U^{-1}(\partial_\nu U)U^{-1}(\partial_\alpha U)U^{-1}(\partial_\beta U)$, but this vanishes, by antisymmetry of $\varepsilon^{\mu\nu\alpha\beta}$ and cyclic symmetry of the trace. Nevertheless, as we will see, there is a lagrangian.

Let us consider a simple problem of the same sort. Consider a particle of mass m constrained to move on an ordinary two-dimensional sphere of radius one. The lagrangian is $\mathfrak{L}=\frac{1}{2}m\int \mathrm{d}t\,\dot{x}_i^2$ and the equation of motion is $m\ddot{x}_i + mx_i(\Sigma_k \dot{x}_k^2)=0$; the constraint is $\Sigma x_i^2 = 1$. This system respects the symmetries $t \leftrightarrow -t$ and separately $x_i \leftrightarrow -x_i$. If we want an equation that is only invariant under the combined operation $t \leftrightarrow -t$, $x_i \leftrightarrow x_i$, the simplest choice is

$$m\ddot{x}_i + mx_i\left(\sum_k \dot{x}_k^2\right) = \alpha\varepsilon_{ijk}x_j\dot{x}_k\,, \tag{5}$$

where α is a constant. To derive this equation from a lagrangian is again troublesome. There is no obvious term whose variation equals the right-hand side (since $\varepsilon_{ijk}x_ix_j\dot{x}_k = 0$).

However, this problem has a well-known solution. The right-hand side of (5) can be understood as the Lorentz force for an electric charge interacting with a magnetic monopole located at the center of the sphere. Introducing a vector potential $\boldsymbol{A}$ such that $\nabla \times \boldsymbol{A} = \boldsymbol{x}/|x|^3$, the action for our problem is

$$I = \int\left(\tfrac{1}{2}m\dot{x}_i^2 + \alpha A_i\dot{x}_i\right)\mathrm{d}t\,. \tag{6}$$

This lagrangian is problematical because A_i contains a Dirac string and certainly does not respect the symmetries of our problem. To explore this quantum mechanically let us consider the simplest form of the Feynman path integral, $\mathrm{Tr}\,\mathrm{e}^{-\beta H} = \int \mathrm{d}x_i(t)\mathrm{e}^{-I}$. In e^{-I} the troublesome term is

$$\exp\left(i\alpha\int_\gamma A_i\,\mathrm{d}x^i\right), \tag{7}$$

where the integration goes over the particle orbit γ: a closed orbit if we discuss the simplest object $\mathrm{Tr}\,\mathrm{e}^{-\beta H}$.

By Gauss's law we can eliminate the vector potential from (7) in favor of the magnetic field. In fact, the closed orbit γ of fig. 1a is the boundary of a disc D, and by Gauss's law we can write (7) in terms of the magnetic flux through D:

$$\exp\left(i\alpha\int_\gamma A_i\,\mathrm{d}x^i\right) = \exp\left(i\alpha\int_{\mathrm{D}} F_{ij}\,\mathrm{d}\Sigma^{ij}\right). \tag{8}$$

The precise mathematical statement here is that since $\pi_1(\mathrm{S}^2)=0$, the circle γ in S^2 is the boundary of a disc D (or more exactly, a mapping γ of a circle into S^2 can be extended to a mapping of a disc into S^2).

The right-hand side of (8) is manifestly well defined, unlike the left-hand side, which suffers from a Dirac string. We could try to use the right-hand side of (8) in a Feynman path integral. There is only one problem: D isn't unique. The curve γ also bounds the disc D′ (fig. 1c). There is no consistent way to decide whether to choose

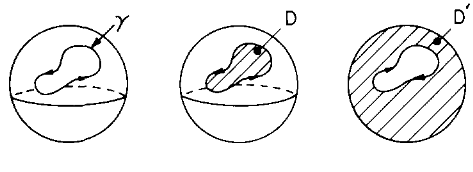

Fig. 1. A particle orbit γ on the two-sphere (part (a)) bounds the discs D (part (b)) and D′ (part (c)).

D or D′ (the curve γ could continuously be looped around the sphere or turned inside out). Working with D′ we would get

$$\exp\left(i\alpha\int_\gamma A_i \mathrm{d}x^i\right) = \exp\left(-i\alpha\int_{D'} F_{ij}\mathrm{d}\Sigma^{ij}\right), \tag{9}$$

where a crucial minus sign on the right-hand side of (9) appears because γ bounds D in a right-hand sense, but bounds D′ in a left-hand sense. If we are to introduce the right-hand side of (8) or (9) in a Feynman path integral, we must require that they be equal. This is equivalent to

$$1 = \exp\left(i\alpha\int_{D+D'} F_{ij}\mathrm{d}\Sigma^{ij}\right). \tag{10}$$

Since D + D′ is the whole two sphere S^2, and $\int_{S^2} F_{ij}\mathrm{d}\Sigma^{ij} = 4\pi$, (10) is obeyed if and only if α is an integer or half-integer. This is Dirac's quantization condition for the product of electric and magnetic charges.

Now let us return to our original problem. We imagine space-time to be a very large four-dimensional sphere M. A given non-linear sigma model field U is a mapping of M into the SU(3) manifold (fig. 2a). Since $\pi_4(\mathrm{SU}(3)) = 0$, the four-sphere in SU(3) defined by U(x) is the boundary of a five-dimensional disc Q.

By analogy with the previous problem, let us try to find some object that can be integrated over Q to define an action functional. On the SU(3) manifold there is a unique fifth rank antisymmetric tensor ω_{ijklm} that is invariant under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$*. Analogous to the right-hand side of eq. (8), we define

$$\Gamma = \int_Q \omega_{ijklm}\mathrm{d}\Sigma^{ijklm}. \tag{11}$$

* Let us first try to define ω at $U = 1$; it can then be extended to the whole SU(3) manifold by an $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ transformation. At $U = 1$, ω must be invariant under the diagonal subgroup of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ that leaves fixed $U = 1$. The tangent space to the SU(3) manifold at $U = 1$ can be identified with the Lie algebra of SU(3). So ω, at $U = 1$, defines a fifth-order antisymmetric invariant in the SU(3) Lie algebra. There is only one such invariant. Given five SU(3) generators A, B, C, D and E, the one such invariant is $\mathrm{Tr}\,ABCDE - \mathrm{Tr}\,BACDE \pm$ permutations. The $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ invariant ω so defined has zero curl ($\partial_i \omega_{jklmn} \pm \text{permutations} = 0$) and for this reason (11) is invariant under infinitesimal variations of Q; there arises only the topological problem discussed in the text.

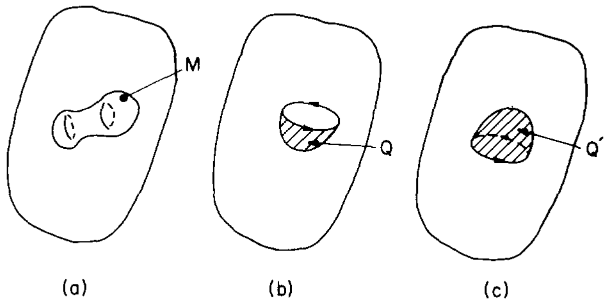

Fig. 2. Space-time, a four-sphere, is mapped into the SU(3) manifold. In part (a), space-time is symbolically denoted as a two sphere. In parts (b) and (c), space-time is reduced to a circle that bounds the discs Q and Q′. The SU(3) manifold is symbolized in these sketches by the interior of the oblong.

As before, we hope to include $\exp(i\Gamma)$ in a Feynman path integral. Again, the problem is that Q is not unique. Our four-sphere M is also the boundary of another five-disc Q′ (fig. 2c). If we let

$$\Gamma' = -\int_{Q'} \omega_{ijklm} \mathrm{d}\Sigma^{ijklm}, \tag{12}$$

(with, again, a minus sign because M bounds Q′ with opposite orientation) then we must require $\exp(i\Gamma) = \exp(i\Gamma')$ or equivalently $\int_{Q+Q'} \omega_{ijklm} \mathrm{d}\Sigma^{ijklm} = 2\pi \cdot \text{integer}$. Since Q + Q′ is a closed five-dimensional sphere, our requirement is

$$\int_{S} \omega_{ijklm} \mathrm{d}\Sigma^{ijklm} = 2\pi \cdot \text{integer},$$

for any five-sphere S in the SU(3) manifold.

We thus need the topological classification of mappings of the five-sphere into SU(3). Since $\pi_5(\mathrm{SU}(3)) = \mathrm{Z}$, every five sphere in SU(3) is topologically a multiple of a basic five sphere S_0. We normalize ω so that

$$\int_{S_0} \omega_{ijklm} \mathrm{d}\Sigma^{ijklm} = 2\pi, \tag{13}$$

and then (with Γ in eq. (11)) we may work with the action

$$I = \tfrac{1}{16} F_\pi^2 \int \mathrm{d}^4x \, \mathrm{Tr}\, \partial_\mu U \, \partial_\mu U^{-1} + n\Gamma, \tag{14}$$

where n is an arbitrary integer. Γ is, in fact, the Wess-Zumino lagrangian. Only the a priori quantization of n is a new result.

The identification of S_0 and the proper normalization of ω is a subtle mathematical problem. The solution involves a factor of two from the Bott periodicity theorem. Without abstract notation, the result [5] can be stated as follows. Let y^i, $i = 1 \ldots 5$ be coordinates for the disc Q. Then on Q (where we need it)

$$\mathrm{d}\Sigma^{ijklm}\,\omega_{ijklm} = -\frac{i}{240\pi^2}\mathrm{d}\Sigma^{ijklm}\left[\mathrm{Tr}\,U^{-1}\frac{\partial U}{\partial y^i}U^{-1}\frac{\partial U}{\partial y^j}U^{-1}\frac{\partial U}{\partial y^k}U^{-1}\frac{\partial U}{\partial y^l}U^{-1}\frac{\partial U}{\partial y^m}\right]. \tag{15}$$

The physical consequences of this can be made more transparent as follows. From eq. (2),

$$U^{-1}\partial_i U = \frac{2i}{F_\pi}\partial_i A + \mathrm{O}(A^2), \qquad \text{where } A = \Sigma\lambda^a\pi^a. \tag{16}$$

So

$$\begin{aligned}\omega_{ijklm}\mathrm{d}\Sigma^{ijklm} &= \frac{2}{15\pi^2F_\pi^5}\mathrm{d}\Sigma^{ijklm}\mathrm{Tr}\,\partial_i A\,\partial_j A\,\partial_k A\,\partial_l A\,\partial_m A + \mathrm{O}(A^6)\\ &= \frac{2}{15\pi^2F_\pi^5}\mathrm{d}\Sigma^{ijklm}\,\partial_i\left(\mathrm{Tr}\,A\,\partial_j A\,\partial_k A\,\partial_l A\,\partial_m A\right) + \mathrm{O}(A^6).\end{aligned}$$

So $\int_Q \omega_{ijklm}\mathrm{d}\Sigma^{ijklm}$ is (to order A^5 and in fact also in higher orders) the integral of a total divergence which can be expressed by Stokes' theorem as an integral over the boundary of Q. By construction, this boundary is precisely space-time. We have, then,

$$n\Gamma = n\frac{2}{15\pi^2F_\pi^5}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,A\,\partial_\mu A\,\partial_\nu A\,\partial_\alpha A\partial_\beta A + \text{higher order terms}. \tag{17}$$

In a hypothetical world of massless kaons and pions, this effective lagrangian rigorously describes the low-energy limit of $K^+K^- \to \pi^+\pi^0\pi^-$ *. We reach the remarkable conclusion that in any theory with SU(3)×SU(3) broken to diagonal SU(3), the low-energy limit of the amplitude for this reaction must be (in units given in (17)) an integer.

What is the value of this integer in QCD? Were n to vanish, the practical interest of our discussion would be greatly reduced. It turns out that if N_c is the number of colors (three in the real world) then $n = N_c$. The simplest way to deduce this is a

* Our formula should agree for $n = 1$ with formulas of ref. [1], as later equations make clear. There appears to be a numerical error on p. 97 of ref. [1] ($\frac{1}{6}$ instead of $\frac{2}{15}$).

procedure that is of interest anyway, viz. coupling to electromagnetism, so as to describe the low-energy dynamics of Goldstone bosons and photons.

Let

$$Q=\begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix}$$

be the usual electric charge matrix of quarks. The functional Γ is invariant under global charge rotations, $U \to U + i\varepsilon[Q, U]$, where ε is a constant. We wish to promote this to a local symmetry, $U \to U + i\varepsilon(x)[Q, U]$, where $\varepsilon(x)$ is an arbitrary function of x. It is necessary, of course, to introduce the photon field A_μ which transforms as $A_\mu \to A_\mu - (1/e)\partial_\mu \varepsilon$; e is the charge of the proton.

Usually a global symmetry can straightforwardly be gauged by replacing derivatives by covariant derivatives, $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$. In the case at hand, Γ is not given as the integral of a manifestly $\mathrm{SU(3)_L \times SU(3)_R}$ invariant expression, so the standard road to gauging global symmetries of Γ is not available. One can still resort to the trial and error Noether method, widely used in supergravity. Under a local charge rotation, one finds $\Gamma \to \Gamma - \int \mathrm{d}^4x\, \partial_\mu \varepsilon J^\mu$ where

$$J^\mu = \frac{1}{48\pi^2}\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\big[Q(\partial_\nu U U^{-1})(\partial_\alpha U U^{-1})(\partial_\beta U U^{-1}) + Q(U^{-1}\partial_\nu U)(U^{-1}\partial_\alpha U)(U^{-1}\partial_\beta U)\big], \tag{18}$$

is the extra term in the electromagnetic current required (from Noether's theorem) due to the addition of Γ to the lagrangian. The first step in the construction of an invariant lagrangian is to add the Noether coupling, $\Gamma \to \Gamma' = \Gamma - e\int \mathrm{d}^4x\, A_\mu J^\mu(x)$. This expression is still not gauge invariant, because J^μ is not, but by trial and error one finds that by adding an extra term one can form a gauge invariant functional

$$\tilde{\Gamma}(U, A_\mu) = \Gamma(U) - e\int \mathrm{d}^4x\, A_\mu J^\mu + \frac{ie^2}{24\pi^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu)A_\alpha \times \mathrm{Tr}\big[Q^2(\partial_\beta U)U^{-1} + Q^2U^{-1}(\partial_\beta U) + QUQU^{-1}(\partial_\beta U)U^{-1}\big]. \tag{19}$$

Our gauge invariant lagrangian will then be

$$\mathcal{L} = \tfrac{1}{16}F_\pi^2\int \mathrm{d}^4x\, \mathrm{Tr}\, D_\mu U D_\mu U^{-1} + n\tilde{\Gamma}. \tag{20}$$

What value of the integer n will reproduce QCD results?

Here we find a surprise. The last term in (18) has a piece that describes $\pi^0 \to \gamma\gamma$. Expanding U and integrating by parts, (18) has a piece

$$A = \frac{ne^2}{48\pi^2 F_\pi} \pi^0 \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} . \tag{21}$$

This agrees with the result from QCD triangle diagrams [6] if $n = N_c$, the number of colors. The Noether coupling $-eA_\mu J^\mu$ describes, among other things, a $\gamma\pi^+\pi^0\pi^-$ vertex

$$B = -\tfrac{2}{3} ie \frac{n}{\pi^2 F_\pi^3} \varepsilon^{\mu\nu\alpha\beta} A_\mu \, \partial_\nu \pi^+ \, \partial_\alpha \pi^- \, \partial_\beta \pi^0 . \tag{22}$$

Again this agrees with calculations [7] based on the QCD VAAA anomaly if $n = N_c$. The effective action $N_c \tilde{\Gamma}$ (first constructed in another way by Wess and Zumino) precisely describes all effects of QCD anomalies in low-energy processes with photons and Goldstone bosons.

It is interesting to try to gauge subgroups of $SU(3)_L \times SU(3)_R$ other than electromagnetism. One may have in mind, for instance, applications to the standard weak interaction model. In general, one may try to gauge an arbitrary subgroup H of $SU(3)_L \times SU(3)_R$, with generators K^σ, $\sigma = 1 \dots r$. Each K^σ is a linear combination of generators T_L^σ and T_R^σ of $SU(3)_L$ and $SU(3)_R$, $K^\sigma = T_L^\sigma + T_R^\sigma$. (Either T_L^σ or T_R^σ may vanish for some values of σ.) For any space-time dependent functions $\varepsilon^\sigma(x)$, let $\varepsilon_L = \Sigma_\sigma T_L^\sigma \varepsilon^\sigma(x)$, $\varepsilon_R = \Sigma_\sigma T_R^\sigma \varepsilon^\sigma(x)$. We want an action with local invariance under $U \to U + i(\varepsilon_L(x) U - U \varepsilon_R(x))$.

Naturally, it is necessary to introduce gauge fields $A_\mu^\sigma(x)$, transforming as $A_\mu^\sigma(x) \to A_\mu^\sigma(x) - (1/e_\sigma)\, \partial_\mu \varepsilon^\sigma + f^{\sigma\tau\rho} \varepsilon^\tau A_\mu^\rho$ where e_σ is the coupling constant corresponding to the generator K^σ, and $f^{\sigma\tau\rho}$ are the structure constants of H. It is useful to define $A_{\mu L} = \Sigma_\sigma e_\sigma A_\mu^\sigma T_L^\sigma$, $A_\mu^R = \Sigma_\sigma e_\sigma A_\mu^\sigma T_R^\sigma$.

We have already seen that Γ incorporates the effects of anomalies, so it is not very surprising that a generalization of Γ that is gauge invariant under H exists only if H is a so-called anomaly-free subgroup of $SU(3)_L \times SU(3)_R$. Specifically, one finds that H can be gauged only if for each σ,

$$\mathrm{Tr}(T_L^\sigma)^3 = \mathrm{Tr}(T_R^\sigma)^3 , \tag{23}$$

which is the usual condition for cancellation of anomalies at the quark level.

If (23) is obeyed, a gauge invariant generalization of Γ can be constructed somewhat tediously by trial and error. It is useful to define $U_{\nu L} = (\partial_\nu U) U^{-1}$ and $U_{\nu R} = U^{-1} \partial_\nu U$. The gauge invariant functional then turns out to be

$$\tilde{\Gamma}(A_\mu, U) = \Gamma(U) + \frac{1}{48\pi^2} \int \mathrm{d}^4 x \, \varepsilon^{\mu\nu\alpha\beta} Z_{\mu\nu\alpha\beta} ,$$

where

$$
\begin{aligned}
Z_{\mu\nu\alpha\beta} = & -\mathrm{Tr}\left[A_{\mu \mathrm{L}} U_{\nu \mathrm{L}} U_{\alpha \mathrm{L}} U_{\beta \mathrm{L}} + (\mathrm{L} \to \mathrm{R})\right] \\
& + i\,\mathrm{Tr}\left[\left[(\partial_\mu A_{\nu \mathrm{L}}) A_{\alpha \mathrm{L}} + A_{\mu \mathrm{L}} (\partial_\nu A_{\alpha \mathrm{L}})\right] U_{\beta \mathrm{L}} + (\mathrm{L} \to \mathrm{R})\right] \\
& + i\,\mathrm{Tr}\left[(\partial_\mu A_{\nu \mathrm{R}}) U^{-1} A_{\alpha \mathrm{L}}\, \partial_\beta U + A_{\mu \mathrm{L}} U^{-1} (\partial_\nu A_{\alpha \mathrm{R}})\, \partial_\beta U\right] \\
& - \tfrac{1}{2} i\,\mathrm{Tr}\left(A_{\mu \mathrm{L}} U_{\nu \mathrm{L}} A_{\alpha \mathrm{L}} U_{\beta \mathrm{L}} - (\mathrm{L} \to \mathrm{R})\right) \\
& + i\,\mathrm{Tr}\left[A_{\mu \mathrm{L}} U A_{\nu \mathrm{R}} U^{-1} U_{\alpha \mathrm{L}} U_{\beta \mathrm{L}} - A_{\mu \mathrm{R}} U^{-1} A_{\nu \mathrm{L}} U U_{\alpha \mathrm{R}} U_{\beta \mathrm{R}}\right] \\
& - \mathrm{Tr}\Big[\left[(\partial_\mu A_{\nu \mathrm{R}}) A_{\alpha \mathrm{R}} + A_{\mu \mathrm{R}} (\partial_\nu A_{\alpha \mathrm{R}})\right] U^{-1} A_{\beta \mathrm{L}} U \\
& \qquad - \left[(\partial_\mu A_{\nu \mathrm{L}}) A_{\alpha \mathrm{L}} + A_{\mu \mathrm{L}} (\partial_\nu A_{\alpha \mathrm{L}})\right] U A_{\beta \mathrm{R}} U^{-1}\Big] \\
& \quad - \mathrm{Tr}\left[A_{\mu \mathrm{R}} U^{-1} A_{\nu \mathrm{L}} U A_{\alpha \mathrm{R}} U_{\beta \mathrm{R}} + A_{\mu \mathrm{L}} U A_{\nu \mathrm{R}} U^{-1} A_{\alpha \mathrm{L}} U_{\beta \mathrm{L}}\right] \\
& \quad - \mathrm{Tr}\left[A_{\mu \mathrm{L}} A_{\nu \mathrm{L}} U (\partial_\alpha A_{\beta \mathrm{R}}) U^{-1} + A_{\mu \mathrm{R}} A_{\nu \mathrm{R}} U^{-1} (\partial_\alpha A_{\beta \mathrm{L}}) U\right] \\
& \quad - i\,\mathrm{Tr}\Big[A_{\mu \mathrm{R}} A_{\nu \mathrm{R}} A_{\alpha \mathrm{R}} U^{-1} A_{\beta \mathrm{L}} U - A_{\mu \mathrm{L}} A_{\nu \mathrm{L}} A_{\alpha \mathrm{L}} U A_{\beta \mathrm{R}} U^{-1} \\
& \qquad + \tfrac{1}{2} A_{\mu \mathrm{L}} A_{\nu \mathrm{L}} U A_{\alpha \mathrm{R}} A_{\beta \mathrm{R}} U^{-1} + \tfrac{1}{2} A_{\mu \mathrm{R}} U^{-1} A_{\nu \mathrm{L}} U A_{\alpha \mathrm{R}} U^{-1} A_{\beta \mathrm{L}} U\Big]. \qquad (24)
\end{aligned}
$$

If eq. (22) for cancellation of anomalies is not obeyed, then the variation of $\tilde{\Gamma}$ under a gauge transformation does not vanish but is

$$
\begin{aligned}
\delta\tilde{\Gamma} = & -\frac{1}{24\pi^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\alpha\beta}\, \mathrm{Tr}\, \varepsilon_{\mathrm{L}}\left[(\partial_\mu A_{\nu \mathrm{L}})(\partial_\alpha A_{\beta \mathrm{L}}) - \tfrac{1}{2} i\, \partial_\mu (A_{\nu \mathrm{L}} A_{\alpha \mathrm{L}} A_{\beta \mathrm{L}})\right] \\
& - (\mathrm{L} \to \mathrm{R}), \qquad (25)
\end{aligned}
$$

in agreement with computations at the quark level [8] of the anomalous variation of the effective action under a gauge transformation.

Thus, Γ incorporates all information usually associated with triangle anomalies, including the restriction on what subgroups H of $\mathrm{SU(3)_L} \times \mathrm{SU(3)_R}$ can be gauged. However, there is another potential obstruction to the ability to gauge a subgroup of $\mathrm{SU(3)_L} \times \mathrm{SU(3)_R}$. This is the non-perturbative anomaly [3] associated with $\pi_4(\mathrm{H})$. Is this anomaly, as well, implicit in Γ? In fact, it is.

Let H be an SU(2) subgroup of $\mathrm{SU(3)_L}$, chosen so that an SU(2) matrix W is embedded in $\mathrm{SU(3)_L}$ as

$$
\hat{W} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{W}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right).
$$

This subgroup is free of triangle anomalies, so the functional $\tilde{\Gamma}$ of eq. (23) is invariant under infinitesimal local H transformations.

However, is $\tilde{\Gamma}$ invariant under H transformations that cannot be reached continuously? Since $\pi_4(\mathrm{SU}(2)) = \mathrm{Z}_2$, there is one non-trivial homotopy class of SU(2) gauge transformations. Let W be an SU(2) gauge transformation in this non-trivial class. Under $\hat{W}$, $\tilde{\Gamma}$ may at most be shifted by a constant, independent of U and A_μ, because $\delta\tilde{\Gamma}/\delta U$ and $\delta\tilde{\Gamma}/\delta A_\mu$ are gauge-covariant local functionals of U and A_μ. Also $\tilde{\Gamma}$ is invariant under $\hat{W}^2$, since $\hat{W}^2$ is equivalent to the identity in $\pi_4(\mathrm{SU}(2))$, and we know $\tilde{\Gamma}$ is invariant under topologically trivial gauge transformations. This does not quite mean that $\tilde{\Gamma}$ is invariant under W. Since $\tilde{\Gamma}$ is only defined modulo 2π, the fact that $\tilde{\Gamma}$ is invariant under W^2 leaves two possibilities for how $\tilde{\Gamma}$ behaves under W. It may be invariant, or it may be shifted by π.

To choose between these alternatives, it is enough to consider a special case. For instance, it suffices to evaluate $\Delta = \tilde{\Gamma}(U=1,\ A_\mu = 0) - \tilde{\Gamma}(U=\hat{W},\ A_\mu = ie^{-1}(\partial_\mu \hat{W})\hat{W}^{-1})$. It is not difficult to see that in this case the complicated terms involving $\varepsilon^{\mu\nu\alpha\beta}Z_{\mu\nu\alpha\beta}$ vanish, so in fact $\Delta = \Gamma(U=1) - \Gamma(U=\hat{W})$. A detailed calculation shows that

$$\Gamma(U=1) - \Gamma(U=\hat{W}) = \pi . \tag{26}$$

This calculation has some other interesting applications and will be described elsewhere [9].

The Feynman path integral, which contains a factor $\exp(iN_c\tilde{\Gamma})$, hence picks up under W a factor $\exp(iN_c\pi) = (-1)^{N_c}$. It is gauge invariant if N_c is even, but not if N_c is odd. This agrees with the determination of the SU(2) anomaly at the quark level [3]. For under H, the right-handed quarks are singlets. The left-handed quarks consist of one singlet and one doublet per color, so the number of doublets equals N_c. The argument of ref. [3] shows at the quark level that the effective action transforms under W as $(-1)^{N_c}$.

Finally, let us make the following remark, which apart from its intrinsic interest will be useful elsewhere [9]. Consider $\mathrm{SU}(3)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{R}$ currents defined at the quark level as

$$J^a_{\mu\mathrm{L}} = \bar{q}\lambda^a\gamma_\mu \tfrac{1}{2}(1-\gamma_5)q , \qquad J^a_{\mu\mathrm{R}} = \bar{q}\lambda^a\gamma_\mu \tfrac{1}{2}(1+\gamma_5)q . \tag{27}$$

By analogy with eq. (17), the proper sigma model description of these currents contains pieces

$$J^{\mu a}_\mathrm{L} = \frac{N_c}{48\pi^2}\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,\lambda^a U_{\nu\mathrm{L}}U_{\alpha\mathrm{L}}U_{\beta\mathrm{L}} ,$$

$$J^{\mu a}_\mathrm{R} = \frac{N_c}{48\pi^2}\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,\lambda^a U_{\nu\mathrm{R}}U_{\alpha\mathrm{R}}U_{\beta\mathrm{R}} , \tag{28}$$

corresponding (via Noether's theorem) to the addition to the lagrangian of $N_c\Gamma$. In this discussion, the λ^a should be traceless SU(3) generators. However, let us try to construct an anomalous baryon number current in the same way. We define the baryon number of a quark (whether left-handed or right-handed) to be $1/N_c$, so that an ordinary baryon made from N_c quarks has baryon number one. Replacing λ^a by $1/N_c$, but including contributions of both left-handed and right-handed quarks, the anomalous baryon-number current would be

$$J^\mu = \frac{1}{24\pi^2}\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\, U^{-1}\partial_\nu U\, U^{-1}\partial_\alpha U\, U^{-1}\partial_\beta U. \tag{29}$$

One way to see that this is the proper, and properly normalized, formula is to consider gauging an arbitrary subgroup not of $SU(3)_L \times SU(3)_R$ but of $SU(3)_L \times SU(3)_R \times U(1)$, U(1) being baryon number. The gauging of U(1) is accomplished by adding a Noether coupling $-eJ^\mu B_\mu$ plus whatever higher-order terms may be required by gauge invariance. (B_μ is a U(1) gauge field which may be coupled as well to some $SU(3)_L \times SU(3)_R$ generator.) With J^μ defined in (29), this leads to a generalization of $\tilde{\Gamma}$ that properly reflects anomalous diagrams involving the baryon-number current (for instance, it properly incorporates the anomaly in the baryon number $SU(2)_L - SU(2)_L$ triangle that leads to baryon non-conservation by instantons in the standard weak interaction model). Eq. (29) may also be extracted from QCD by methods of Goldstone and Wilczek [10].

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